Quiz 10; Wednesday, November 1
MATH 110 with Professor Stankova
DSP
GSI: Saad Qadeer

Solutions

You have 10 minutes to complete the quiz. Calculators are not permitted. Please include all relevant calculations and explanations (unless stated otherwise).

1. (12 points) Let T be a linear operator on a finite-dimensional vector space V. Prove that if $\text{nullity}(T^m) = \text{nullity}(T^{m+1})$ for some positive integer m, then $\ker(T^m) = \ker(T^{m+k})$ for any integer $k \geq 0$.

   Note first that $\ker(T^m) \subseteq \ker(T^{m+1})$ along with $\text{nullity}(T^m) = \text{nullity}(T^{m+1})$ shows that $\ker(T^m) = \ker(T^{m+1})$.

   We prove the claim by induction on k. For $k = 0$, the result obviously holds. Suppose it is true for some k, that is, $\ker(T^m) = \ker(T^{m+k})$. Let $v \in \ker(T^{m+k+1}) \Rightarrow T^{m+k+1}(v) = 0 \Rightarrow T^{m+1}(T^k(v)) = 0 \Rightarrow T^k(v) \in \ker(T^{m+1})$. As $\ker(T^{m+1}) = \ker(T^m)$, we have $T^m(T^k(v)) = 0 \Rightarrow T^{m+k}(v) = 0$ so $v \in \ker(T^{m+k}) = \ker(T^m)$ by the inductive hypothesis. This establishes $\ker(T^{m+k+1}) \subseteq \ker(T^m)$. Along with $\ker(T^m) \subseteq \ker(T^{m+k+1})$, it allows us to conclude that $\ker(T^m) = \ker(T^{m+k+1})$, proving the result.

2. (1 + 1 + 1 points) Mark each statement as True or False. You do not need to show your work but a blank answer is worth 0 points and an incorrect answer is worth -1 point.

   (a) Every eigenvector of a linear operator T is also a generalized eigenvector.

   True: If v is an eigenvector of T, then $(T - \lambda I)(v) = 0$.

   (b) Let T be a linear operator on an n-dimensional vector space V. Then, $\mathrm{Im}(T^n) = \mathrm{Im}(T^{n+1})$.

   True: The stabilizing exponent never exceeds n.

   (c) Jordan blocks of the same size but corresponding to different eigenvalues may not commute.

   False: They always commute as long as they have the same size.